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A note on operator probability theory involving numerical ranges ☆

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ABSTRACT

In this note, the relationships between the expectation and variance in operator probability theory and numerical range of operators are considered.

1. Introduction

The probability theory on operator algebras such as C^*-algebras and von Neumann algebras has also been called noncommutative probability or quantum probability theory. In recent years, a lot of work has been devoted to noncommutative probability theory (see [1–3,5,7–9] and references therein). In this note, from an interest of operator theorists, we shall only consider the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. So the probability theory on $\mathcal{B}(\mathcal{H})$ is called operator probability theory which is due to Gudder [8].

Throughout this paper, let $\mathcal{H}$ be a complex Hilbert space with the inner product $(\cdot,\cdot)$, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, T^*, $\sigma(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the adjoint, the spectrum, the null space and the range of T, respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $T = T^*$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$.

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For an operator $A \in \mathcal{B}(\mathcal{H})$, the numerical range $\mathcal{W}(A)$ of A is defined by

$$\mathcal{W}(A) := \{(Ax, x) : x \in \mathcal{H}, (x, x) = 1\}. \tag{1}$$

As is well known, $\sigma(A) \subseteq \overline{\mathcal{W}(A)}$, where $\overline{K}$ denotes the closure of K. By the well-known Toeplitz–Hausdorff theorem, the numerical range $W(A)$ of an operator A is a convex subset of $\mathbb{C}$ and it satisfies the so-called spectral inclusion property

$$\sigma_p(A) \subset W(A), \quad \sigma(A) \subset \overline{W(A)},$$

where $\sigma_p(T)$ denotes the point spectrum of $T \in \mathcal{B}(\mathcal{H})(see[12])$.

Let $\mathcal{D}(\mathcal{H})$ denote the set of density operators which are positive trace class operators on $\mathcal{H}$ with unit trace. $\rho \in \mathcal{D}(\mathcal{H})$ is called a state. For a state ρ, there exists a set $\{x_i\}_{i=1}^k$ of orthonormal vectors in $\mathcal{H}$ such that

$$\rho = \sum_{i=1}^{k} \lambda_i x_i \otimes x_i,$$

where $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i > 0$, $1 \leqslant i \leqslant k$, k is finite or infinite. If $A \in \mathcal{B}(\mathcal{H})$ and ρ is a state, the ρ-expectation $E_\rho(A)$ of A is defined [8] by

$$E_\rho(A) = \mathrm{tr}(\rho A). \tag{2}$$

If $\rho_x = x \otimes x$ is a pure state for a unit vector $x \in \mathcal{H}$, then

$$E_{\rho_x}(A) = \mathrm{tr}(\rho_x A) = (Ax, x). \tag{3}$$

The ρ-moments of A are $\mathrm{tr}(\rho A^n)$, $n = 1, 2, \ldots$, and the ρ-variance of A is

$$\mathrm{Var}_\rho(A) = E_\rho[(A - E_\rho(A)I)^2] = E_\rho(A^2) - E_\rho(A)^2. \tag{4}$$

The ρ-absolute variance of A is

$$|\ \mathrm{Var}_\rho\ |\ (A) = E_\rho[|\ A - E_\rho(A)I\ |^2] = E_\rho(|\ A\ |^2) - |\ E_\rho(A)\ |^2\ . \tag{5}$$

For $A, B \in \mathcal{B}(\mathcal{H})$, the ρ-correlation coefficient $\mathrm{Cor}_\rho(A, B)$ of A, B is defined by

$$\begin{aligned} \mathrm{Cor}_\rho(A, B) &= E_\rho[(A - E_\rho(A)I)^*(B - E_\rho(B)I)] \\ &= E_\rho(A^*B) - E_\rho(A^*)E_\rho(B). \end{aligned} \tag{6}$$

From (3) and the definitions of ρ-expectation (2), ρ-variance (4), ρ-absolute variance (5) and ρ-correlation coefficient (6), it is evident that there exist close relations among numerical range and these concepts involving operators. In the following, we shall investigate the ρ-expectation, ρ-variance, ρ-absolute variance and ρ-correlation coefficient dependent on numerical ranges.

Firstly, recall two known results concerning with the product of numerical ranges of operators which are used hereafter.

Theorem G-N (see [8]). Let A, B, C be bounded self-adjoint operators on a complex Hilbert space $\mathcal{H}$ with the property

$$(Ax, x)(Bx, x) = (Cx, x)$$

for all $x \in \mathcal{H}$ with $\parallel x \parallel = 1$. Then $A = cI$ or $B = cI$ for some $c \in \mathbb{R}$.

More recently, we have gotten a generalization of Theorem G-N as follows.

Theorem D-D-Z-S (see [3]). Let A, B and $C \in \mathcal{B}(\mathcal{H})$. Then

$$(Ax, x)(Bx, x) = (Cx, x) \tag{7}$$

for all $x \in \mathcal{H}$ with $\| x \| = 1$ if and only if there exists a complex number $c \in \mathbb{C}$ such that $A = cI$ and $C = cB$ or $B = cI$ and $C = cA$.

For the sake of convenience, we now present some lemmas which will be used later on.

Lemma 1.1 (see [8]). *For $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$,*

(1) $E_\rho(A - \lambda I) = E_\rho(A) - \lambda$.
(2) $Var_\rho(A - \lambda I) = Var_\rho(A)$.
(3) $| Var_\rho | (A - \lambda I) = | Var_\rho | (A)$.
(4) *If A is self-adjoint, then $Var_\rho(A) = | Var_\rho | (A)$.*

Lemma 1.2 (see [8]). *For $A \in \mathcal{B}(\mathcal{H})$, then*

$$E_\rho(A^*) = \overline{E_\rho(A)}.$$

This paper is organized as follows. In Section 2, we shall discuss the ρ-expectations and numerical ranges of operators. The ρ-variances and maximum numerical ranges of operators will be considered in Section 3. In Section 4, we shall study the ρ-variances and ρ-absolute variances of operators by using numerical ranges of operators and propose an open question on ρ-absolute variances of operators. In Section 5, we shall investigate the ρ-correlation coefficients and numerical ranges of operators. In particular, the construction characterization of two uncorrelated operators (see [8]) is obtained.

2. ρ-Expectations and numerical ranges

From (3), we have $E_{\rho_x}(A) = (Ax, x) \in W(A)$ for a unit vector $x \in \mathcal{H}$. This shows us that there exist some close relations between ρ-expectations and numerical ranges of operators. In fact, in this section, the main aim is to establish the exact connection between the numerical range $W(A)$ of A and the set of all $E_\rho(A)$ for $\rho \in \mathcal{D}(\mathcal{H})$.

We begin with a lemma.

Lemma 2.1. *For $A \in \mathcal{B}(\mathcal{H})$, we have*

$$W(A + \lambda I) = \lambda + W(A)$$

and

$$W(e^{i\theta}A) = e^{i\theta}W(A).$$

In the following, we present the main result of this section.

Theorem 2.2. *For $A \in \mathcal{B}(\mathcal{H})$, we have $\{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\} = W(A)$.*

Proof. Firstly, we see that $\{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\}$ is a convex set in $\mathbb{C}$ since $\mathcal{D}(\mathcal{H})$ is a convex set of $\mathcal{B}(\mathcal{H})$. Hence $W(A) \subseteq \{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\}$ because $(Ax, x) = E_{\rho_x}(A)$ for a unit vector $x \in \mathcal{H}$.

In general, for a state $\rho \in \mathcal{D}(\mathcal{H})$, there exist a set $\{x_i\}_i^k$ of orthonormal vectors and a set $\{\lambda_i\}$ of positive numbers with $\sum_{i=1}^k \lambda_i = 1$ such that $\rho = \sum_{i=1}^k \lambda_i x_i \otimes x_i$, where k is finite or infinite. In this case, it is clear that

$$E_\rho(A) = \mathrm{tr}(\rho A) = \mathrm{tr}\left(\sum_{i=1}^k \lambda_i x_i \otimes x_i A\right) = \sum_{i=1}^k \lambda_i (Ax_i, x_i) \in \overline{W(A)}.$$

This shows that $\{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\} \subseteq \overline{W(A)}$. So,

$$W(A) \subseteq \{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\} \subseteq \overline{W(A)}. \tag{8}$$

From (8), to prove that $W(A) = \{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\}$, it is sufficient to show that $\alpha \in \partial W(A)$ and $\alpha \notin W(A)$ implies that $\alpha \notin \{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\}$.

On the contrary, assume that there exists an α_0 such that $\alpha_0 \in \partial W(A)$, $\alpha_0 \notin W(A)$ and $\alpha_0 \in \{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\}$. Without loss of generality, by Lemma 2.1 we can suppose that $\alpha_0 > 0$, a support line of $W(A)$ throughout α_0 is a vertical line and $W(A)$ lies in the left half-plane separated by the support line. In this case, there exists a state $\rho_0 = \sum_{i=1}^{k_0} \lambda_i y_i \otimes y_i$, where $\{y_i\}_{i=1}^{k_0}$ is a set of orthonormal vectors, $\lambda_i > 0$, $1 \leqslant i \leqslant k_0$ and $\sum_{i=1}^{k_0} \lambda_i = 1$, such that

$$\alpha_0 = \mathrm{tr}(\rho_0 A) = \sum_{i=1}^{k_0} \lambda_i (Ay_i, y_i). \tag{9}$$

If $A = B + iC$ is the Cartesian decomposition of A, where B and C are self-adjoint, then from (9) we have

$$\alpha_0 = \sum_{i=1}^{k_0} \lambda_i (Ay_i, y_i) = \sum_{i=1}^{k_0} \lambda_i (By_i, y_i) + i \sum_{i=1}^{k_0} \lambda_i (Cy_i, y_i). \tag{10}$$

Observing that (By_i, y_i) and (Cy_i, y_i), $1 \leqslant i \leqslant k_0$, are real numbers. Comparing the two sides of (10), we get

$$\begin{cases} \alpha_0 = \sum_{i=1}^{k_0} \lambda_i (By_i, y_i), \\ \sum_{i=1}^{k_0} \lambda_i (Cy_i, y_i) = 0. \end{cases} \tag{11}$$

Moreover, note that $\sum_{i=1}^{k_0} \lambda_i = 1$, $\lambda_i > 0$, $1 \leqslant i \leqslant k_0$ and $(By_i, y_i) \leqslant \alpha_0$. We will have $(By_i, y_i) = \alpha_0$, $1 \leqslant i \leqslant k_0$. First, if there exists an i_0, $1 \leqslant i_0 \leqslant k_0$, with $(Cy_{i_0}, y_{i_0}) = 0$, then $\alpha_0 = (By_{i_0}, y_{i_0}) = (By_{i_0}, y_{i_0}) + (Cy_{i_0}, y_{i_0}) = (Ay_{i_0}, y_{i_0}) \in W(A)$. It is a contradiction. Second, if $(Cy_i, y_i) \neq 0$, $1 \leqslant i \leqslant k_0$, since $\sum_{i=1}^{k_0} \lambda_i (Cy_i, y_i) = 0$ and $\lambda_i > 0$, $1 \leqslant i \leqslant k_0$, then there should be $1 \leqslant i_1, i_2 \leqslant k_0$, such that $(Cy_{i_1}, y_{i_1}) > 0$ and $(Cy_{i_2}, y_{i_2}) < 0$. So $(Ay_{i_1}, y_{i_1}) = (By_{i_1}, y_{i_1}) + i(Cy_{i_1}, y_{i_1}) = \alpha_0 + i(Cy_{i_1}, y_{i_1}) \in W(A)$ and $(Ay_{i_2}, y_{i_2}) = (By_{i_2}, y_{i_2}) + i(Cy_{i_2}, y_{i_2}) = \alpha_0 + i(Cy_{i_2}, y_{i_2}) \in W(A)$. Denote $\beta = \frac{-(Cy_{i_2}, y_{i_2})}{(Cy_{i_1}, y_{i_1}) - (Cy_{i_2}, y_{i_2})}$. It is clear that $0 < \beta < 1$. Now, we have

$$\alpha_0 = \beta (Ay_{i_1}, y_{i_1}) + (1 - \beta)(Ay_{i_2}, y_{i_2}).$$

This means that $\alpha_0 \in W(A)$. It is also a contradiction.

Finally, we obtain that

$$\{E_\rho(A) : \rho \in \mathcal{D}(\mathcal{H})\} = W(A). \quad \square$$

Corollary 2.3. *For $A \in \mathcal{B}(\mathcal{H})$, $\sup\{| E_\rho(A) | : \rho \in \mathcal{D}(\mathcal{H})\} = \omega(A)$, where $\omega(A)$ denotes the numerical radius of A.*

3. ρ-Variances and maximum numerical ranges

In this section, we consider the ρ-variances and maximum numerical ranges of operators. We first recall the definition of maximum numerical range of an operator. For an operator $K \in \mathcal{B}(\mathcal{H})$, the maximum numerical range $W_0(K)$ (see [10]) of K is defined by

$$W_0(K) = \{\lambda : \lambda = \lim_n (Kx_n, x_n), \lim_n \| Kx_n \| = \| K \|, \| x_n \| = 1, \quad x_n \in \mathcal{H}\}.$$

For maximum numerical ranges of operators, it is well known that

Lemma 3.1 (see [10]). *For* $A \in \mathcal{B}(\mathcal{H})$, $\inf\{\| A - \lambda I \|: \lambda \in \mathbb{C}\} = \| A - \lambda_0 I \|$ *if and only if* $0 \in W_0(A - \lambda_0 I)$.

In general, $W(A)$ is not necessarily closed, but the maximum numerical range $W_0(A)$ of an operator $A \in \mathcal{B}(\mathcal{H})$ is always a closed convex set of the complex plane and $W_0(A) \subseteq \overline{W(A)}$ (see [10]). So there should always exist a complex number λ_0 such that $\inf\{\| A - \lambda I \|: \lambda \in \mathbb{C}\} = \| A - \lambda_0 I \|$ by Lemma 3.1. It is easy to see that the λ_0 satisfying $\inf\{\| A - \lambda I \|: \lambda \in \mathbb{C}\} = \| A - \lambda_0 I \|$ is unique.

More recently, Chen et al. (see [2]) have showed that

Proposition 3.2 [2]. *For* $A \in \mathcal{B}(\mathcal{H})$ *and* $\rho_x = x \otimes x$ *with a unit vector* $x \in \mathcal{H}$, *we have*

$$\sup\{| Var_{\rho_x} | (A) : x \in \mathcal{H}, \| x \| = 1\} = \inf\{\| A - \lambda I \|^2: \lambda \in \mathbb{C}\} = \| A - \lambda_0 I \|^2$$

for some $\lambda_0 \in \mathbb{C}$.

Moreover, we have

Theorem 3.3. *For* $A \in \mathcal{B}(\mathcal{H})$, *we have* $\sup\{| Var_\rho | (A) : \rho \in \mathcal{D}(\mathcal{H})\} = \inf\{\| A - \lambda I \|^2: \lambda \in \mathbb{C}\} = \| A - \lambda_0 I \|^2$ *for some* $\lambda_0 \in \mathbb{C}$.

Proof. If $\rho \in \mathcal{D}(\mathcal{H})$, then there exists a set $\{x_i\}_{i=1}^k$ of orthonormal vectors such that

$$\rho = \sum_{i=1}^{k} \lambda_i x_i \otimes x_i$$

with $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i > 0$, where k is finite or infinite. In this case, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} & | \mathrm{Var}_\rho | (A) \\ = & | \mathrm{Var}_\rho | (A - \lambda I) \\ = & \textstyle\sum_{i=1}^k \lambda_i \| (A - \lambda I)x_i \|^2 - | \sum_{i=1}^k \lambda_i (Ax_i, x_i)|^2 \\ \leqslant & \| A - \lambda I \|^2 . \end{aligned}$$

Since $\lambda \in \mathbb{C}$ is arbitrary, we have

$$| \mathrm{Var}_\rho | (A) \leqslant \inf\{\| A - \lambda I \|^2: \lambda \in \mathbb{C}\} = \| A - \lambda_0 I \|^2$$

for some $\lambda_0 \in \mathbb{C}$. Hence,

$$\sup\{| \mathrm{Var}_\rho | (A) : \rho \in \mathcal{D}(\mathcal{H})\} \leqslant \| A - \lambda_0 I \|^2 .$$

On the other hand, for λ_0 with $0 \in W_0(A - \lambda_0 I)$, we have

$$\| A - \lambda_0 I \| = \min\{\| A - \lambda I \|: \lambda \in \mathbb{C}\}$$

by Lemma 3.1 and there exists a sequence $\{y_i\}$ of unit vectors such that

$$\lim_i \| (A - \lambda_0 I)y_i \| = \| A - \lambda_0 I \|, \lim_i ((A - \lambda_0 I)y_i . y_i) = 0.$$

Therefore

$$\| A - \lambda_0 I \|^2 = \sup\{| \mathrm{Var}_{\rho_{y_i}} | (A) : i = 1, 2, \ldots\}.$$

In general,

$$\sup\{\mid \mathrm{Var}_{\rho_{y_i}} \mid (A) : i = 1, 2, \ldots\} \leqslant \sup\{\mid \mathrm{Var}_\rho \mid (A) : \rho \in \mathcal{D}(\mathcal{H})\}.$$

Hence

$$\| A - \lambda_0 I \|^2 \leqslant \sup\{\mid \mathrm{Var}_\rho \mid (A) : \rho \in \mathcal{D}(\mathcal{H})\}.$$

Finally, we have

$$\sup\{\mid \mathrm{Var}_\rho \mid (A) : \rho \in \mathcal{D}(\mathcal{H})\} = \| A - \lambda_0 I \|^2 . \quad \square$$

4. ρ-Variances and ρ-absolute variances

From Lemma 1.1, if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\mathrm{Var}_\rho(A) = \mid \mathrm{Var}_\rho \mid (A)$ for all $\rho \in \mathcal{D}(\mathcal{H})$. In this section, we shall remove the restriction of that operator is self-adjoint, giving a general conclusion as Theorem 4.1. At the end of this section, we shall also propose an open question about ρ-absolute variances of operators.

Theorem 4.1. *For $A \in \mathcal{B}(\mathcal{H})$, let $A = B + iC$ be the Cartesian decomposition of A. Then $\mathrm{Var}_\rho(A) = \mid \mathrm{Var}_\rho \mid (A)$ for all $\rho \in \mathcal{D}(\mathcal{H})$ if and only if $C = cI$ for some real number $c \in \mathbb{R}$.*

Proof. " $\Rightarrow$ ". Denote $\rho_x = x \otimes x$ for a unit vector $x \in \mathcal{H}$. By the assumption that $\mathrm{Var}_\rho(A) = \mid \mathrm{Var}_\rho \mid (A)$ for all $\rho \in \mathcal{D}(\mathcal{H})$, we specially have $\mathrm{Var}_{\rho_x}(A) = \mid \mathrm{Var}_{\rho_x} \mid (A)$ for all unit vectors $x \in \mathcal{H}$. That is,

$$E_{\rho_x}(A^2) - E_{\rho_x}(A)^2 = E_{\rho_x}(\mid A \mid^2) - \mid E_{\rho_x}(A) \mid^2 .$$

Hence,

$$(A^2x, x) - (Ax, x)^2 = (\mid A \mid^2 x, x) - \mid (Ax, x) \mid^2 . \tag{12}$$

Therefore,

$$\begin{aligned} &(A^2x, x) - (Ax, x)^2 \\ &= ((B^2 - C^2 + i(BC + CB))x, x) - (Bx, x)^2 + (Cx, x)^2 - 2i(Bx, x)(Cx, x) \\ &= (B^2x, x) - (C^2x, x) - (Bx, x)^2 + (Cx, x)^2 + i((BC + CB)x, x) - 2(Bx, x)(Cx, x). \end{aligned}$$

Observing that $(\mid A \mid^2 x, x) - \mid (Ax, x) \mid^2$ is real, compare the imaginary parties of the two sides of (12), we have

$$((BC + CB)x, x) - 2(Bx, x)(Cx, x) = 0.$$

Therefore

$$((BC + CB)x, x) = 2(Bx, x)(Cx, x).$$

Since x is an arbitrary unit vector, by Theorem G-N, $B = cI$ or $C = cI$ for some real number c.

If $C = cI$, it is nothing to do.

If $B = bI$ for a real number b, then $iC = A - bI$. By Lemma 1.1 and (12), we have

$$((iC)^2x, x) - (iCx, x)^2 = (\mid iC \mid^2 x, x) - \mid (iCx, x) \mid^2 .$$

So

$$-(C^2x, x) + (Cx, x)^2 = (C^2x, x) - (Cx, x)^2.$$

Hence,

$$(Cx, x)^2 = (C^2x, x).$$

By Theorem G-N again, there exists a real number c such that $C = cI$.

" $\Leftarrow$ ". By Lemma 1.1, if $C = cI$, then $\text{Var}_\rho(A) = \text{Var}_\rho(A - icI) = \text{Var}_\rho(B) = \mid \text{Var}_\rho \mid (B) = \mid \text{Var}_\rho \mid (B + icI) = \mid \text{Var}_\rho \mid (A)$. $\square$

Corollary 4.2. *For $A \in \mathcal{B}(\mathcal{H})$, $Var_\rho(A) = \mid Var_\rho \mid (A)$ for all $\rho \in \mathcal{D}(\mathcal{H})$ if and only if there exist a complex number $a \in \mathbb{C}$ and a self-adjoint operator A_0 such that $A = A_0 + aI$.*

Theorem 4.3. *If $A \in \mathcal{B}(\mathcal{H})$ has the Cartesian decomposition $A = B + iC$, then A is a normal operator if and only if*

$$\mid Var_\rho \mid (A) = Var_\rho(B) + Var_\rho(C)$$

for all $\rho \in \mathcal{D}$.

Proof

" $\Leftarrow$ " For a pure state ρ_x with a unit vector x, by the assumption that $\mid \text{Var}_\rho \mid (A) = \text{Var}_\rho(B) + \text{Var}_\rho(C)$, we also have $\mid \text{Var}_{\rho_x} \mid (A) = \text{Var}_{\rho_x}(B) + \text{Var}_{\rho_x}(C)$, that is,

$$(A^*Ax, x) - (A^*x, x)(Ax, x) = (B^2x, x) - (Bx, x)^2 + (C^2x, x) - (Cx, x)^2. \tag{13}$$

Since $(A^*Ax, x) - (A^*x, x)(Ax, x) = ((B^2 + C^2 + i(BC - CB))x, x) - (Bx, x)^2 - (Cx, x)^2$, from (13), we get

$$((BC - CB)x, x) = 0$$

for any unit vector $x \in \mathcal{H}$. This shows that $BC - CB = 0$. That is, $BC = CB$. Therefore, A is normal.

" $\Rightarrow$ " If $A = B + iC$ is normal, then $BC = CB$. Hence, $A^*A = B^2 + C^2$. In this case, for $\rho \in \mathcal{D}$, we have

$$\begin{aligned}
\mid \text{Var}_\rho \mid (A) &= E_\rho(\mid A \mid^2) - \mid E_\rho(A) \mid^2 \\
&= E_\rho(B^2 + C^2) - E_\rho(B - iC)E_\rho(B + iC) \\
&= \text{tr}(\rho(B^2 + C^2)) - \text{tr}(\rho(B - iC))\text{tr}(\rho(B + iC)) \\
&= \text{tr}(\rho B^2) - (\text{tr}(\rho B))^2 + \text{tr}(\rho C^2) - (\text{tr}(\rho C))^2 \\
&= E_\rho(B^2) - E_\rho(B)^2 + E_\rho(C^2) - E_\rho(C)^2 \\
&= \text{Var}_\rho(B) + \text{Var}_\rho(C). \square
\end{aligned}$$

By Lemma 1.1, we obtain another version of Theorem 4.3.

Theorem 4.4. *If $A \in \mathcal{B}(\mathcal{H})$ has the Cartesian decomposition $A = B + iC$, then A is a normal operator if and only if*

$$\mid Var_\rho \mid (A) = \mid Var_\rho \mid (B) + \mid Var_\rho \mid (C)$$

for all $\rho \in \mathcal{D}(\mathcal{H})$.

From Theorem 4.4, a natural question is raised as follows.

Open question 4.5. Which conditions for operators A and B are necessary and sufficient such that

$$|\operatorname{Var}_\rho|(A+B) = |\operatorname{Var}_\rho|(A) + |\operatorname{Var}_\rho|(B)$$

for all $\rho \in \mathcal{D}(\mathcal{H})$?

In general, for a pure state ρ_x, we have

$$\begin{aligned}
& |\operatorname{Var}_{\rho_x}|(A+B) \\
= & ((A^*+B^*)(A+B)x, x) - ((A^*+B^*)x, x)((A+B)x, x) \\
= & (A^*Ax, x) + (A^*Bx, x) + (B^*Ax, x) + (B^*Bx, x) \\
& -(A^*x, x)(Ax, x) - (A^*x, x)(Bx, x) - (B^*x, x)(Ax, x) - (B^*x, x)(Bx, x) \\
= & |\operatorname{Var}_{\rho_x}|(A) + |\operatorname{Var}_{\rho_x}|(B) \\
& +(A^*Bx, x) + (B^*Ax, x) - (A^*x, x)(Bx, x) - (B^*x, x)(Ax, x).
\end{aligned}$$

If $|\operatorname{Var}_\rho|(A+B) = |\operatorname{Var}_\rho|(A) + |\operatorname{Var}_\rho|(B)$ for all $\rho \in \mathcal{D}$, we should have

$$(A^*Bx, x) + (B^*Ax, x) = (A^*x, x)(Bx, x) + (B^*x, x)(Ax, x), \tag{14}$$

for all unit vectors $x \in \mathcal{B}(\mathcal{H})$. If (14) holds for all unit vectors $x \in \mathcal{B}(\mathcal{H})$, what can we say about A and B?

5. ρ-Correlation coefficients and products of numerical ranges

From formulas (4), (5) and (6), it is easy to check that

$$\operatorname{Cor}_\rho(A + aI, B + \beta I) = \operatorname{Cor}_\rho(A, B),$$

$$\operatorname{Cor}_\rho(A, B) = \overline{\operatorname{Cor}_\rho(B, A)} \tag{15}$$

and

$$|\operatorname{Var}_\rho|(A+B) = |\operatorname{Var}_\rho|(A) + |\operatorname{Var}_\rho|(B) + 2Re(\operatorname{Cor}_\rho(A, B)).$$

If $A = B$, then

$$\operatorname{Cor}_\rho(A, A) = E_\rho(A^*A) - E_\rho(A^*)E_\rho(A) = |\operatorname{Var}_\rho|(A).$$

In this section, we shall give some other results about ρ-correlation coefficient.

Let us begin with a lemma.

Lemma 5.1. *If A is a normal operator, then $\|A\| = \omega(A)$.*

Proposition 5.2. *For $A, B \in \mathcal{B}(\mathcal{H})$, $Cor_\rho(A, B) = 0$ for any $\rho \in D(H)$ if and only if $A = cI$ or $B = cI$ for some $c \in \mathbb{C}$.*

Proof. Let x be a unit vector. By the assumption, we have

$$\begin{aligned}
\operatorname{Cor}_{\rho_x}(A, B) &= E_{\rho_x}(A^*B) - E_{\rho_x}(A^*)E_{\rho_x}(B) \\
&= (A^*Bx, x) - (A^*x, x)(Bx, x) = 0.
\end{aligned}$$

That is,

$$(A^*Bx, x) = (A^*x, x)(Bx, x), \quad \text{for} \quad x \in \mathcal{H} \text{ with } \| x \| = 1.$$

By Theorem D-D-Z-S, at least one of A and B is a scalar multiple of the identity.

Conversely, if $A = cI$ or $B = cI$ for some $c \in \mathbb{C}$, it is clear that $\mathrm{Cor}_\rho(A, B) = 0$ for any $\rho \in D(H)$. $\square$

Remark. Proposition 5.2 gives a precise characterization that two operators are uncorrelated (see [8]). It is a generalization of the statement that operators A and B are uncorrelated if and only if $E_\rho(A^*B) = E_\rho(A^*)E_\rho(B)$ which is due to Gudder [8].

From (15), we see that $\mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A)$ is a pure imaginary number for any pair A and B of operators. If A and B are self-adjoint, the following may be interesting.

Theorem 5.3. *If A and B are self-adjoint, then*

$$\sup\{| \, Cor_\rho(A, B) - Cor_\rho(B, A) \, |: \rho \in \mathcal{D}(\mathcal{H})\} = \| AB - BA \| .$$

Proof. On the one hand, for $\rho \in \mathcal{D}(\mathcal{H})$, ρ has the representation

$$\rho = \sum_i \lambda_i x_i \otimes x_i,$$

where $\lambda_i > 0$, $\{x_i\}$ is a set of orthonormal vectors in $\mathcal{H}$, and $\sum_i \lambda_i = 1$. Directly calculating,

$$\begin{aligned}
&| \, \mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A) \, | \\
&= | \, E_\rho(A^*B) - E_\rho(A^*)E_\rho(B) - E_\rho(B^*A) + E_\rho(B^*)E_\rho(A) \, | \\
&= | \, E_\rho(AB) - E_\rho(A)E_\rho(B) - E_\rho(BA) + E_\rho(B)E_\rho(A) \, | \\
&= | \, E_\rho(AB) - E_\rho(BA) \, | \\
&= | \, \textstyle\sum_i \lambda_i((AB - BA)x_i, x_i) \, | \\
&\leqslant \| AB - BA \| .
\end{aligned}$$

Moreover, because that $\rho \in \mathcal{D}(\mathcal{H})$ is arbitrary, we have

$$\sup\{| \, \mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A) \, |: \rho \in \mathcal{D}(\mathcal{H})\} \leqslant \| AB - BA \| .$$

On the other hand, it is clear that $AB - BA$ is a normal operator since A and B are self-adjoint, hence there exists a sequence $\{y_j\}$ of unit vectors such that $\| AB - BA \| = \lim_{j \to \infty} | \, ((AB - BA)y_j, y_j) \, |$ by Lemma 5.1. Define ρ_{y_j} by

$$\rho_{y_j} = y_j \otimes y_j.$$

We have

$$| \, \mathrm{Cor}_{\rho_{y_j}}(A, B) - \mathrm{Cor}_{\rho_{y_j}}(B, A) \, | = | \, ((AB - BA)y_j, y_j) \, | \to \| AB - BA \| \ (\text{as } j \to \infty).$$

That is,

$$\sup\{| \, \mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A) \, |: \rho \in \mathcal{D}(\mathcal{H})\} \geqslant \| AB - BA \| .$$

Finally, combining the two cases above, we get

$$\sup\{| \, \mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A) \, |: \rho \in \mathcal{D}(\mathcal{H})\} = \| AB - BA \| . \quad \square$$

In fact, Theorem 5.3 gives a characterization of the norm of the commutator of two self-adjoint operators by using the ρ-correlation coefficients. Moreover, we have

Corollary 5.4. *If A and B are self-adjoint, then $Cor_\rho(A, B) \in \mathbb{R}$ for all $\rho \in \mathcal{D}(\mathcal{H})$ if and only if $AB = BA$.*

Proof. Let A and B are self-adjoint. Observing that $\mathrm{Cor}_\rho(A, B) \in \mathbb{R}$ if and only if $\mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A) = 0$ for $\rho \in \mathcal{D}(\mathcal{H})$, it is obvious by Theorem 5.3. □

If we consider the spectra of A and B, we obtain

Corollary 5.5. *If A and B are self-adjoint, then*

$$\sup\{|\ Cor_\rho(A, B) - Cor_\rho(B, A)\ |: \rho \in \mathcal{D}(\mathcal{H})\} \leqslant \frac{1}{2}(\alpha_1 - \alpha_2)(\beta_1 - \beta_2),$$

where $\alpha_1 = \max\{\alpha \in \sigma(A)\}$, $\alpha_2 = \min\{\alpha \in \sigma(A)\}$, $\beta_1 = \max\{\beta \in \sigma(B)\}$ and $\beta_2 = \min\{\beta \in \sigma(B)\}$.

From the corollary 7 in [11] or [4] Corollary 5.5, is clear.

As the end of this paper, from Proposition 5.2, Theorem 5.3 and Corollary 5.4, we have the following question.

Remark 5.6. What can we say about $\sup\{|\ \mathrm{Cor}_\rho(A, B) - \mathrm{Cor}_\rho(B, A)\ |: \rho \in \mathcal{D}(\mathcal{H})\}$ for any pair A and B of operators in $\mathcal{B}(\mathcal{H})$?

References

[1] A. Arias, A. Gheondea, S. Gudder, Fixed points of quantum operations, J. Math. Phys., 43 (2002) 5872–5881.
[2] Z.L. Chen, H.X. Cao, H.K. Du, Some properties of quantum probability, Chinese J. Engrg. Math. 27 (1) (2010) 1669–1671.
[3] H.K. Du, Y.N. Dou, H.Y. Zhang, L.M. Shen, A generalization of Gudder–Nagy's theorem with numerical ranges of operators, J. Math. Phys. 52 (2) (2011) 023501.
[4] F. Kittaneh, Inequalities for commutators of positive operators, J. Funct. Anal. 205 (2007) 132–143.
[5] Y. Li, Fixed point of dual quantum operation, J. Math. Anal. Appl. 282 (2011) 172–179.
[6] S. Gudder, Sequential quantum measurements, Rep. Math. Phys. 60 (2007) 273–288.
[7] S. Gudder, Fuzzy quantum probability theory, preprint, AIP Conference Proc. Foundation of Probability and Physics, vol. 750, 2004, pp. 114–121.
[8] S. Gudder, Operator probability theory, Int. J. Pure Appl. Math. 39 (2007) 511–526.
[9] S. Gudder, G. Nagy, Sequential quantum measurements, J. Math. Phys. 42 (2001) 5212–5222.
[10] J.G. Stampfli, The norm of a derivation, Pac. J. Math. 33 (1970) 737–747.
[11] Y.Q. Wang, H.K. Du, Norms of commutators of self-adjoint operators, J. Math. Anal. Appl. 342 (2008) 747–751.
[12] K.Z. Wang, P.Y. Wu, Numerical ranges of weighted shifts, J. Math. Anal. Appl. 381 (2011) 897–909.